Some Theorems on Hamiltonian Graphs

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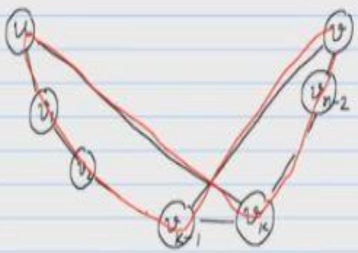
Dirac's theorem :

Theorem

If G is a simple graph with $n\ (\geq 3)$ vertices and if $d(v) \geq \frac{n}{2}$ for each vertex v, then G is Hamiltonian.

proof By contradiction

Let G be such a counter example to the theorem so that no graphs on n vertices with more edges than G is also a counter example. Let u and v be two non-adjacent vertices of G. Then there is a Hamiltonian path joining u and v in G.



Let $d(u) = k \geq \frac{n}{2}$.

Now we prove that if u is adjacent to v_k then v_{k-1} cannot be adjacent to v.

If possible, let v be adjacent to v_{k-1}.

Then we have the Hamiltonian cycle

$v\, v_{n-2} \cdots v_k\, u\, v_1\, v_2 \cdots v_{k-1}\, v$.

Since we assumed that G is not Hamiltonian, this is not possible.

So v is not adjacent to v_{k-1}.

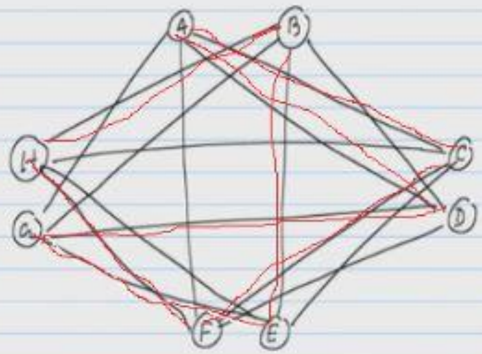
So v is not adjacent to at least k of the $n-1$ vertices.

Thus $d(v) \leq n-1-k \leq n-1-\frac{n}{2} = \frac{n}{2}-1$

This is a contradiction; thus G must be Hamiltonian.

Example on Dirac's theorem :

Ex. According to Dirac's theorem the following graph contain a Hamiltonian cycle. Find it.



G

$d(v) = 4 \geq \frac{n}{2} = \frac{8}{2} = 4$

$|V| = 8$

hamiltonian Cycle

A C F H B E G D A

Bondy's theorem :

Theorem (Bondy)

Let G be a simple graph on $n\ (\geq 3)$ vertices.

Suppose $u, v \in V$ and such that $(u,v) \notin E$ and

$d(u) + d(v) \geq n$. In this case

G is Hamiltonian iff $G + (u,v)$ is Hamiltonian.

proof $\Rightarrow$ G is Hamiltonian then obviously $G+(u,v)$ is Hamiltonian.

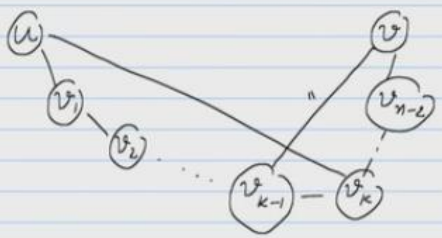
$\Leftarrow$ Given that $G + (u,v)$ is Hamiltonian

we need to prove that G is Hamiltonian.

Suppose G is not Hamiltonian.

There is a Hamiltonian path / spanning path in G

joining u and v.



If u is adjacent to v_k, then v cannot be adjacent to v_{k-1}. Let $d(u) = k$.

So v is not adjacent to atleast k of the $n-1$ vertices.

Thus $d(v) \leq n-1-k$

$\therefore$ $d(u) + d(v) \leq n-1-k+k = n-1$

$\leq n-1$

which is a contradiction.

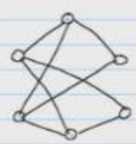
So G is Hamiltonian.

Closure of a Graph

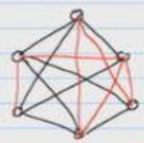
CLOSURE OF A GRAPH

The closure $C(G)$ of a graph of order n (i.e. total vertices=n) is obtained from G by recursively joining pair of non-adjacent vertices whose degree sum is atleast n until no such pair exists.

Example

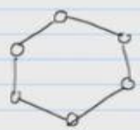


G $\quad$ $n = 6$



$C(G)$ $\quad K_6$

As degree sum of each red connected edges should be n=6



$G = C_6$

here the graph itself is iits closure becoz. sum of degree is less than n=6

Theorem: Closure of a Graph Vs. Hamiltonian Graph

Th: A graph is Hamiltonian iff its closure is Hamiltonian.

proof: G is Hamiltonian iff $G + (u,v)$ is Hamiltonian.

$d(u) + d(v) \geq n$

(becoz. of closure of graph property)

To establish a graph is Hamiltonian, it is sufficient if we show that its closure is complete.

If the closure of a graph is complete

(Closure of graph is always complete as we know)

graph then the closure is Hamiltonian.

So G is Hamiltonian.